Exercise 3

Solve the differential equation.

$$y'' + 2y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2 = 0$$

Solve for r.

$$r = \{-i\sqrt{2}, i\sqrt{2}\}$$

Two solutions to the ODE are $e^{-i\sqrt{2}x}$ and $e^{i\sqrt{2}x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{-i\sqrt{2}x} + C_2 e^{i\sqrt{2}x} \\ &= C_1(\cos\sqrt{2}x - i\sin\sqrt{2}x) + C_2(\cos\sqrt{2}x + i\sin\sqrt{2}x) \\ &= (C_1 + C_2)\cos\sqrt{2}x + (-iC_1 + iC_2)\sin\sqrt{2}x \\ &= C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}x, \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.